

A NUMBER FIELD EXTENSION OF A QUESTION OF MILNOR

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To Professor Ram Murty on the occasion of his sixtieth birthday

ABSTRACT. Milnor [7] formulated a conjecture about rational linear independence of some special Hurwitz zeta values. In [3], this conjecture was studied and an extension of Milnor's conjecture was suggested. In this note, we investigate the number field generalisation of this extended Milnor conjecture. We indicate the motivation for considering this number field case by noting that such a phenomenon is true in an analogous context. We also study some new spaces related to normalised Hurwitz zeta values.

1. INTRODUCTION

For a real number x with $0 < x \leq 1$ and $s \in \mathbb{C}$ with $\Re(s) > 1$, the Hurwitz zeta function is defined by

$$\zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

This (as a function of s) can be analytically extended to the entire complex plane except at $s = 1$ where it has a simple pole with residue one. Note that $\zeta(s, 1) = \zeta(s)$ is the classical Riemann zeta function.

In 1983, Milnor (see [7], §6) made a conjecture about the linear independence of certain special Hurwitz zeta values over \mathbb{Q} . More precisely, he suggested the following:

For integers $q, k > 1$, the \mathbb{Q} -linear space $V(k, \mathbb{Q})$ generated by the real numbers

$$\zeta(k, a/q), \quad 1 \leq a < q \text{ with } (a, q) = 1$$

has dimension $\varphi(q)$.

The relevance of these Hurwitz zeta values is that they form a natural generating set for the study of special values of Dirichlet series associated to periodic arithmetic functions. More precisely, one is interested in the special values of L -series of the form

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

where f is defined over integers and $f(n+q) = f(n)$ for all integers n with a fixed modulus q . Typically, f takes algebraic values. Running over arithmetic progressions mod q , one immediately deduces that

$$L(s, f) = q^{-s} \sum_{a=1}^q f(a) \zeta(s, a/q).$$

In [3], the second and the third author with M. Ram Murty studied Milnor's conjecture and derived a non-trivial lower bound for the dimension of $V(k, \mathbb{Q})$, namely that the dimension is at least half of the conjectured dimension. They also obtained a conditional improvement of this lower bound and noted that any unconditional improvement of this "half" threshold will have remarkable consequences in relation to irrationality of the numbers $\zeta(2d+1)/\pi^{2d+1}$.

Furthermore in [3], the authors suggested a generalisation of the original conjecture of Milnor. There are at least two reasons for considering such a generalisation. First is that the inhomogeneous version of Baker's theorem for linear forms in logarithms of algebraic numbers naturally suggests such a generalisation. Secondly, typically one is interested in irrationality of $\zeta(2d+1)/\pi^{2d+1}$ as well as that of $\zeta(2d+1)$ and this generalisation predicts the irrationality of both these numbers. Following is this extension suggested by the authors (see [3]):

Extended Milnor conjecture: *In addition to the original Milnor's conjecture, $V(k, \mathbb{Q}) \cap \mathbb{Q} = \{0\}$.*

In an earlier work [2], the first author considered various ramifications of this conjecture.

In this work, we investigate the number field extension of the above conjecture. One of the reasons for considering such an extension is that we are interested in the transcendence of odd zeta values $\zeta(2d+1)$ as well as of the normalised values $\zeta(2d+1)/\pi^{2d+1}$. This extension predicts such an eventuality. Moreover, there is a related set up where the analogous statement can be established unconditionally. This is the content of Theorem 2.2 in the next section. See also [4] and [6] for a modular interpretation of the conjectural transcendence of the normalised values $\zeta(2d+1)/\pi^{2d+1}$.

It will be evident that considering the extended Milnor conjecture to a number field \mathbb{K} comes with a caveat, namely it depends on the arithmetic of \mathbb{K} (for instance compare Theorem 3.3 with Corollary 4.2).

As we shall see in section 3, the expected \mathbb{K} -dimension is $\varphi(q)$ for number fields \mathbb{K} such that $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. In such cases, the mathematics is somewhat amenable and one can derive similar lower bounds for these dimensions as has been done in the earlier works [3] and [5].

On the other hand, when the ambient number field \mathbb{K} has non-trivial intersection with the q -th cyclotomic field $\mathbb{Q}(\zeta_q)$, nothing is known. In section 4, we investigate this difficult case and derive some results. We also try to highlight the crux of the complexity.

Finally in the last section, we consider some new spaces generated by normalised Hurwitz zeta values which appear naturally in the study of irrationality of odd zeta values. The mathematics in this set up is somewhat different. For instance, the parity of k enters into the question non-trivially which is not evident in the earlier questions.

2. THE ANALOGOUS CASE FOR THE SPACE GENERATED BY THE VALUES OF $L(1, \chi)$

In this section, we consider the question of linear independence of the special values $L(1, \chi)$ as χ runs over non-trivial Dirichlet characters mod q . This serves as a guiding line for the questions addressed in this work.

One of reasons why we have a clearer picture in this context is the following seminal theorem of Baker (see [1], also [10]).

Theorem 2.1. *If $\alpha_1, \dots, \alpha_n$ are non-zero algebraic numbers such that the numbers $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over rationals, then the numbers $1, \log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

In an earlier work, R. Murty and K. Murty [9] used Ramachandra units to prove that the values $L(1, \chi)$ as χ runs through non-trivial even Dirichlet characters mod q are linearly independent over $\overline{\mathbb{Q}}$. We note that without much effort, the following extension of their result can be obtained.

Theorem 2.2. *The numbers $L(1, \chi)$ as χ runs through non-trivial even Dirichlet characters mod q and 1 are linearly independent over $\overline{\mathbb{Q}}$.*

Proof. As noticed in [9], each of these special values is a linear form in logarithms involving real multiplicatively independent units of Ramachandra. Thus any linear combination

$$\sum_{\substack{\chi \text{ even} \\ \chi \neq 1}} \lambda_\chi L(1, \chi)$$

with λ_χ algebraic, not all zero, is necessarily transcendental by Baker's theorem. \square

We now highlight as well as summarise the salient features in this set up. This will serve as an indicator of what to expect in the more involved case of special values related to Milnor's conjecture.

- When χ is an odd character, it can be seen that $L(1, \chi)$ is an algebraic multiple of π (see page 38 of [13] for instance). Thus the $L(1, \chi)$ values when χ runs through odd characters mod q form a one dimensional vector space over $\overline{\mathbb{Q}}$. Let us call this space the arithmetic space and denote it by V_{ar} . Since π is transcendental, we have

$$V_{ar} \cap \overline{\mathbb{Q}} = \{0\}.$$

- The $\overline{\mathbb{Q}}$ vector space generated by the $L(1, \chi)$ values when χ runs through non trivial even characters mod q is of optimal dimension $\varphi(q)/2 - 1$. Let us call this space the transcendental space and denote it by V_{tr} . If we assume Schanuel's conjecture, all these values are algebraically independent. Recall that Schanuel's conjecture (see [10], page 111) is the assertion that for any collection of complex numbers $\alpha_1, \dots, \alpha_n$ that are linearly independent over \mathbb{Q} , the transcendence degree of the field

$$\mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n})$$

over \mathbb{Q} is at least n .

- The transcendental space intersects $\overline{\mathbb{Q}}$ trivially, that is,

$$V_{tr} \cap \overline{\mathbb{Q}} = \{0\}.$$

This follows from Theorem 2.2.

- Finally, we can prove the following stronger assertion, namely that the following sum

$$V_{ar} + V_{tr} + \overline{\mathbb{Q}}$$

is direct.

Here is a proof of this assertion. The values of $L(1, \chi)$ for non-trivial even characters χ are linear forms in logarithms of real positive algebraic numbers. On the other hand, when χ is an odd character, $L(1, \chi)$ is an algebraic multiple of $\log(-1)$. By Baker's theorem, any $\overline{\mathbb{Q}}$ -relation involving logarithms of positive real algebraic numbers (from non-trivial even characters) and $\log(-1)$ will result in a \mathbb{Z} -linear relation between these numbers. This will lead to a contradiction as $\log(-1) = i\pi$ is purely imaginary. This along with Theorem 2.2 proves that the above sum is direct.

3. GENERALISED MILNOR CONJECTURE OVER NUMBER FIELDS INTERSECTING $\mathbb{Q}(\zeta_q)$ TRIVIAALLY

Let us first set some notations in relation to the extended Milnor conjecture over number fields. Let \mathbb{K} be a number field and $k > 1, q > 2$ be

integers. Let $\widehat{V}_k(q, \mathbb{K})$ be the \mathbb{K} -linear space generated by the numbers

$$1, \zeta(k, a/q), \quad 1 \leq a < q \text{ with } (a, q) = 1.$$

We are interested in the dimension of this space. This as we shall see will depend on the chosen number field \mathbb{K} . We first isolate the following two canonical subspaces of $\widehat{V}_k(q, \mathbb{K})$, namely the \mathbb{K} -linear space spanned by the following sets of real numbers:

$$\{\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) : (a, q) = 1, 1 \leq a < q/2\},$$

which we refer to as the “arithmetic space” and the space spanned by

$$\{\zeta(k, a/q) + (-1)^{k+1} \zeta(k, 1 - a/q) : (a, q) = 1, 1 \leq a < q/2\}$$

which we call the “transcendental space”. Let us denote them by $V_{ar}(\mathbb{K})$ and $V_{tr}(\mathbb{K})$ respectively.

We now state the following results which are of relevance in this set up. First, one has the following theorem of Okada [11] (see also [8]).

Lemma 3.1. *Let k and q be positive integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representations mod q such that the union $T \cup (-T)$ constitutes a complete set of co-prime residue classes mod q . Let \mathbb{K} be a number field such that $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then the set of real numbers*

$$\frac{d^{k-1}}{dz^{k-1}} \cot(\pi z)|_{z=a/q}, \quad a \in T$$

is linearly independent over \mathbb{K} .

We shall be frequently using the following identity (see [8], for instance):

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z)|_{z=a/q}. \quad (1)$$

Finally, one has the following result established in [5]:

Lemma 3.2. *For any $1 \leq a < q/2$ with $(a, q) = 1$, the number*

$$\frac{\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)}{(i\pi)^k}$$

lies in the q -th cyclotomic field $\mathbb{Q}(\zeta_q)$.

Now one can see that each generating element of the arithmetic space $V_{ar}(\mathbb{K})$ is actually transcendental. However, we call the space $V_{ar}(\mathbb{K})$ arithmetic as it still generates a one-dimensional space over $\overline{\mathbb{Q}}$. This follows from Lemma 3.2.

On the other hand, one expects all the generating elements of the transcendental space $V_{tr}(\mathbb{K})$ to be algebraically independent and hence of dimension $\varphi(q)/2$ over $\overline{\mathbb{Q}}$. Note that the results of the previous section supports such an expectation.

Having fixed these notations, we now consider the relatively accessible case, namely when $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. In this case, we can prove the following lower bound for the dimension of $\widehat{V}_k(q, \mathbb{K})$.

Theorem 3.3. *Let $k > 1, q > 2$ be positive integers and \mathbb{K} be a number field with $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$. Then*

$$\dim_{\mathbb{K}} \widehat{V}_k(q, \mathbb{K}) \geq \frac{\varphi(q)}{2} + 1.$$

Proof. By Lemma 3.1, the following $\varphi(q)/2$ numbers

$$\frac{d^{k-1}}{dz^{k-1}}(\pi \cot \pi z)|_{z=a/q}, \quad 1 \leq a < q/2, \quad (a, q) = 1$$

are linearly independent over \mathbb{K} since \mathbb{K} intersects $\mathbb{Q}(\zeta_q)$ trivially. Further, by Lemma 3.2, each of these numbers

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)$$

is an algebraic multiple of π^k and hence $V_{ar}(\mathbb{K})$ does not contain 1. Thus using the identity given by (1), we have the lower bound mentioned in the theorem. \square

Any improvement of the above lower bound for odd k will have remarkable consequences. In particular, we have the following consequence which is not difficult to derive.

Proposition 3.4. *Let $k > 1$ be an odd integer. If $\dim_{\mathbb{K}} \widehat{V}_k(4, \mathbb{K}) = 3$ for all real number fields \mathbb{K} , then $\zeta(k)$ is transcendental.*

In this context, we have the following conditional improvement of the above lower bound for odd k .

Theorem 3.5. *Let $k > 1$ be an odd integer and $q, r > 2$ be two co-prime integers. Also, let \mathbb{K} be a real number field with discriminant $d_{\mathbb{K}}$ co-prime to qr . Assume that $\zeta(k) \notin \mathbb{K}$. Then either*

$$\begin{aligned} \dim_{\mathbb{K}} \widehat{V}_k(q, \mathbb{K}) &\geq \frac{\varphi(q)}{2} + 2 \\ \text{or} \quad \dim_{\mathbb{K}} \widehat{V}_k(r, \mathbb{K}) &\geq \frac{\varphi(r)}{2} + 2. \end{aligned}$$

Proof. Suppose not. Then by the above theorem, we have

$$\dim_{\mathbb{K}} \widehat{V}_k(q, \mathbb{K}) = \frac{\varphi(q)}{2} + 1$$

and

$$\dim_{\mathbb{K}} \widehat{V}_k(r, \mathbb{K}) = \frac{\varphi(r)}{2} + 1.$$

Now for the first case, the numbers

$$1, \zeta(k, a/q) - \zeta(k, 1 - a/q), \text{ where } (a, q) = 1, 1 \leq a < q/2$$

generate $\widehat{V}_k(q, \mathbb{K})$ over \mathbb{K} . Since k is odd, we have

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{(\pi i)^k} \in \mathbb{Q}(\zeta_q) \subseteq \mathbb{K}(\zeta_q). \quad (2)$$

Now consider the identity

$$\zeta(k) \prod_{\substack{p \text{ prime,} \\ p|q}} (1 - p^{-k}) = q^{-k} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \zeta(k, a/q) \in \widehat{V}_k(q, \mathbb{K}).$$

Thus $\zeta(k) \in \widehat{V}_k(q, \mathbb{K})$ and hence

$$\zeta(k) = \alpha_1 + \sum_{\substack{(a,q)=1 \\ 1 \leq a < q/2}} \beta_a [\zeta(k, a/q) - \zeta(k, 1 - a/q)]$$

for some $\alpha_1, \beta_a \in \mathbb{K}$. Using (2)

$$a_1 := \frac{\zeta(k) - \alpha_1}{i\pi^k} \in \mathbb{K}(\zeta_q).$$

Similarly,

$$\dim_{\mathbb{K}} \widehat{V}_k(r, \mathbb{K}) = \frac{\varphi(r)}{2} + 1$$

implies

$$a_2 := \frac{\zeta(k) - \alpha_2}{i\pi^k} \in \mathbb{K}(\zeta_r) \quad (3)$$

with $\alpha_2 \in \mathbb{K}$. Thus,

$$a_1 i\pi^k + \alpha_1 = a_2 i\pi^k + \alpha_2$$

which implies

$$(a_1 - a_2) i\pi^k = \alpha_2 - \alpha_1.$$

Transcendence of π implies that $\alpha_1 = \alpha_2$, $a_1 = a_2$ and hence

$$\frac{\zeta(k) - \alpha_1}{i\pi^k} \in \mathbb{K}(\zeta_q) \cap \mathbb{K}(\zeta_r) = \mathbb{K}$$

because $(d_{\mathbb{K}}, qr) = 1$. Since $\mathbb{K} \subset \mathbb{R}$, $\zeta(k) = \alpha_1 \in \mathbb{K}$, a contradiction. This completes the proof of the theorem. \square

We end the section by proposing what we believe should be the extended Milnor conjecture for number fields \mathbb{K} that intersect $\mathbb{Q}(\zeta_q)$ trivially, namely:

The dimension of the \mathbb{K} -linear space $\widehat{V}_k(q, \mathbb{K})$ when $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}$ is equal to $\varphi(q) + 1$.

When \mathbb{K} intersects $\mathbb{Q}(\zeta_q)$ non-trivially, the situation is more involved and this is the content of the next section.

4. EXTENDED MILNOR CONJECTURE OVER NUMBER FIELDS INTERSECTING $\mathbb{Q}(\zeta_q)$ NON-TRIVIALY

In this section, we consider the case when the ambient number field \mathbb{K} intersects $\mathbb{Q}(\zeta_q)$ non-trivially. The difficulty here is that the result of Okada is no longer valid which precludes us from concluding about the dimension of the arithmetic space $V_{ar}(\mathbb{K})$.

Here we have the following theorem.

Theorem 4.1. *Let $k > 1, q > 2$ be integers. For $1 \leq a < q/2$, $(a, q) = 1$, let λ_a be defined as*

$$\lambda_a := \frac{\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)}{(\pi i)^k}.$$

If $\lambda_a \in \mathbb{K}$ for some a as above, then

$$2 \leq \dim \widehat{V}_k(q, \mathbb{K}) \leq \frac{\varphi(q)}{2} + 2.$$

Proof. We first recall that (see [3])

$$\frac{\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)}{(\pi i)^k} = A \sum_{b=1}^q (\zeta_q^{ab} + (-1)^k \zeta_q^{-ab}) B_k(b/q)$$

where $B_k(x)$ is the k -th Bernoulli polynomial and A is a rational number. Suppose $\lambda_a \in \mathbb{K}$. Then $\lambda_a \in K := \mathbb{K} \cap \mathbb{Q}(\zeta_q)$. Since K is Galois (in fact abelian) over \mathbb{Q} , every element of the Galois group $G = \text{Gal}(\mathbb{Q}(\zeta_q)/\mathbb{Q})$ when restricted to K gives an automorphism of K . Note that for any $(r, q) = 1$, the corresponding element σ_r of G , given by the action $\zeta_q \rightarrow \zeta_q^r$, takes λ_a to λ_{ar} . Hence $\lambda_c \in K$ for all $(c, q) = 1$ with $1 \leq c < q/2$. Now the upper bound is obvious as $V_{ar}(\mathbb{K})$ is of dimension one over \mathbb{K} and because $1 \notin V_{ar}(\mathbb{K})$. This also gives the lower bound. \square

As a corollary, we have

Corollary 4.2. *For $k > 1, q > 2$, we have*

$$2 \leq \dim \widehat{V}_k(q, \mathbb{Q}(\zeta_q)) \leq \frac{\varphi(q)}{2} + 2.$$

To get an idea of the difficulty, we now give an instance where the dimension of $V_{ar}(\mathbb{K})$ does not go down even when \mathbb{K} intersects $\mathbb{Q}(\zeta_q)$ non-trivially.

Theorem 4.3. *Let $k > 1, q > 2$ be a natural number and $\mathbb{K} = \mathbb{Q}(i\sqrt{d})$ for some square-free natural number $d \geq 1$. If $\mathbb{K} \cap \mathbb{Q}(\zeta_q) = \mathbb{Q}(i\sqrt{d})$, then*

$$\dim_{\mathbb{K}} V_{ar}(\mathbb{K}) = \varphi(q)/2$$

and thus

$$\dim_{\mathbb{K}} \widehat{V}_k(q, \mathbb{K}) \geq \frac{\varphi(q)}{2} + 1.$$

Proof. Write

$$\lambda'_a := \zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q),$$

where $(a, q) = 1$ with $1 \leq a < q/2$. If these numbers are linearly dependent over \mathbb{K} , then

$$\sum_a (\alpha_a + i\sqrt{d} \beta_a) \lambda'_a = 0,$$

where α_a, β_a are rational numbers. Since by Okada's theorem the numbers λ'_a 's are linearly independent over \mathbb{Q} , we have $\alpha_a = 0 = \beta_a$ for all such a . Then the theorem follows by noticing that $\pi^k \notin \overline{\mathbb{Q}}$. \square

As indicated earlier, the dimension of the space $\widehat{V}_k(q, \mathbb{K})$ for odd k is particularly important. Here one has the following proposition.

Proposition 4.4. *There exists an integer $q_0 > 2$ such that for all integers $q > 2$ with $(q_0, q) = 1$, the dimension of the space $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ is at least 3 for infinitely many odd k .*

Proof. Suppose that for any two co-prime integers q and r , we have

$$\dim \widehat{V}_k(q, \mathbb{Q}(\zeta_q)) = 2 \text{ and } \dim \widehat{V}_k(r, \mathbb{Q}(\zeta_r)) = 2.$$

As k is an odd integer, we have

$$\zeta(k, a/q) - \zeta(k, 1 - a/q) \in i\pi^k \mathbb{Q}(\zeta_q)$$

for all $1 \leq a < q/2$ with $(a, q) = 1$ and

$$\zeta(k, b/r) - \zeta(k, 1 - b/r) \in i\pi^k \mathbb{Q}(\zeta_r)$$

for all $1 \leq b < r/2$ with $(b, r) = 1$. Hence the spaces $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ and $\widehat{V}_k(r, \mathbb{Q}(\zeta_r))$ are generated by 1 and $i\pi^k$ over $\mathbb{Q}(\zeta_q)$ and $\mathbb{Q}(\zeta_r)$ respectively.

Again we know that $\zeta(k)$ belongs to both the spaces $\widehat{V}_k(q, \mathbb{Q}(\zeta_q))$ and $\widehat{V}_k(r, \mathbb{Q}(\zeta_r))$. Hence $\zeta(k)$ can be written as

$$\zeta(k) = \alpha_1 + \alpha_2 i\pi^k = \beta_1 + \beta_2 i\pi^k \tag{4}$$

for some $\alpha_1, \alpha_2 \in \mathbb{Q}(\zeta_q)$ and $\beta_1, \beta_2 \in \mathbb{Q}(\zeta_r)$. Thus we have

$$(\alpha_2 - \beta_2) i\pi^k = \beta_1 - \alpha_1.$$

Transcendence of π implies that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. As $\mathbb{Q}(\zeta_q) \cap \mathbb{Q}(\zeta_r) = \mathbb{Q}$, we see that both α_1, α_2 are rational numbers. Then by (4), it follows that $\zeta(k)$ is necessarily rational. By the work of Rivoal [12], we know that there are infinitely many odd k such that $\zeta(k)$ is irrational. Thus we have the proposition. \square

We summarise the issues involved in the number field version of the extended Milnor conjecture. This is modelled upon our experience in relation to the corresponding questions involving the interrelation among the values of $L(1, \chi)$ as discussed in Section 2.

- It is clear that $V_{ar}(\mathbb{K}) \cap \overline{\mathbb{Q}} = \{0\}$. However the dimension of $V_{ar}(\mathbb{K})$ over \mathbb{K} is most likely the only parameter which depends on the ambient number field \mathbb{K} . As we noticed, $V_{ar}(\mathbb{K})$ is a one-dimensional vector space over $\overline{\mathbb{Q}}$. The dimension of the arithmetic space does not seem to have any transcendental input.
- One expects that the elements of the generating set of $V_{tr}(\mathbb{K})$ are linearly independent over \mathbb{K} and therefore have dimension $\varphi(q)/2$. In fact, one expects this to hold even over $\overline{\mathbb{Q}}$. This is likely to be a transcendental issue.

- One believes that

$$V_{tr}(\mathbb{K}) \cap \mathbb{K} = 0.$$

Again, this is likely to be a transcendental issue.

- Finally, one expects that the sum

$$V_{tr}(\mathbb{K}) + V_{ar}(\mathbb{K}) + \mathbb{K}$$

is direct. But this supposedly involves the question of independence between families of different transcendental numbers and hence may have both transcendental as well as arithmetic input.

5. SPACE GENERATED BY NORMALISED HURWITZ ZETA VALUES

In this section, we define the following new class of \mathbb{Q} -linear spaces.

Definition 5.1. *For integers $k > 1, q > 2$, let $S_k(q)$ be the \mathbb{Q} -linear space defined by*

$$S_k(q) := \mathbb{Q} - \text{span of } \left\{ \frac{\zeta(k, a/q)}{\pi^k} : 1 \leq a < q, (a, q) = 1 \right\}$$

and $\widehat{S}_k(q)$ be the \mathbb{Q} -linear space defined by

$$\widehat{S}_k(q) := \mathbb{Q} - \text{span of } \left\{ 1, \frac{\zeta(k, a/q)}{\pi^k} : 1 \leq a < q, (a, q) = 1 \right\}.$$

These spaces appear similar to the spaces related to Milnor and extended Milnor conjecture respectively. But there is an important distinction, namely the parity of k enters the picture. Recall, the conjectural dimension of the extended Milnor spaces is independent of parity of k . But this is no longer the case for these new spaces.

However as before, in relation to these spaces also, we can deduce the following lower bound.

Theorem 5.2. *Let $k > 1$ and $q > 2$ be two integers. Then*

$$\dim_{\mathbb{Q}} S_k(q) \geq \frac{\varphi(q)}{2}.$$

Proof. First note that the space $S_k(q)$ is also spanned by the following sets of real numbers:

$$\left\{ \frac{\zeta(k, a/q) + \zeta(k, 1 - a/q)}{\pi^k} \mid (a, q) = 1, 1 \leq a < q/2 \right\},$$

$$\left\{ \frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{\pi^k} \mid (a, q) = 1, 1 \leq a < q/2 \right\}.$$

Then, again by the following ubiquitous identity

$$\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q) = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (\pi \cot \pi z) \Big|_{z=a/q}$$

and by the result of Okada, the numbers on the right hand side for $1 \leq a < q/2$ with $(a, q) = 1$ are \mathbb{Q} -linearly independent. Hence the following numbers

$$\frac{\zeta(k, a/q) + (-1)^k \zeta(k, 1 - a/q)}{\pi^k}, \quad 1 \leq a < q/2, (a, q) = 1$$

are linearly independent over \mathbb{Q} . □

Interestingly, the parity of k enters the picture non-trivially as seen by the following proposition.

Theorem 5.3. *Let $k > 1$ be an even integer and $q > 2$ be any integer. Then $S_k(q) = \widehat{S}_k(q)$.*

Proof. Note that for even k ,

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \frac{\zeta(k, a/q)}{\pi^k} = q^k \prod_{\substack{p \text{ prime,} \\ p|q}} (1 - p^{-k}) \frac{\zeta(k)}{\pi^k} \in \mathbb{Q}.$$

Hence for k even,

$$\mathbb{Q} \subset S_k(q)$$

and thus $S_k(q) = \widehat{S}_k(q)$. \square

Thus for an even k , \mathbb{Q} lies in the associated normalised arithmetic space. However, when k is an odd integer, we expect the picture to be different. For instance, unlike the earlier case, \mathbb{Q} does not seem to belong to the normalised arithmetic space, at least when $4 \nmid q$.

Theorem 5.4. *Let $k > 1$ be an odd integer and $4 \nmid q$. Then \mathbb{Q} does not belong to the normalised arithmetic space, that is, the \mathbb{Q} -vector space generated by the numbers*

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{\pi^k}, \quad 1 \leq a < q/2, (a, q) = 1.$$

intersects \mathbb{Q} trivially.

Proof. Suppose that 1 belongs to the given space. As noted before, each of these numbers

$$\frac{\zeta(k, a/q) - \zeta(k, 1 - a/q)}{\pi^k}, \quad 1 \leq a < q/2, (a, q) = 1$$

when multiplied by i lie inside the q -th cyclotomic field. Therefore, if 1 is expressible as a rational linear combination of these numbers, then i necessarily lies in the q th cyclotomic field. This not possible as $4 \nmid q$. This completes the proof. \square

Further, when k is odd, we can also derive the following result by employing the earlier techniques as in Proposition 4.4.

Theorem 5.5. *Let $k > 1$ be an odd integer. Then there exists a $q_0 > 2$ such that*

$$\dim_{\mathbb{Q}} S_k(q) \geq \frac{\varphi(q)}{2} + 1$$

for any $q > 2$ co-prime to q_0 .

To conclude, while

$$S_k(q) = \widehat{S}_k(q)$$

when k is even, there is reason to believe that

$$S_k(q) \subsetneq \widehat{S}_k(q)$$

when k is odd, at least when $4 \nmid q$.

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